INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2005; **47**:825–831
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.846

Performance of numerical methods on the non-unique solution to the Riemann problem for the shallow water equations

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SUMMARY

For certain initial conditions, the exact solution to the Riemann problem for the shallow water equations is not unique. We test the performance of several numerical methods on such initial data and establish that the numerical solution can pick out different exact solutions. Moreover, the numerical solution does not necessarily converge towards the picked-out exact solution. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: shallow water equations; Riemann problem; non-strictly hyperbolic non-conservative system; test cases

1. INTRODUCTION

In this work, we are concerned with the system of one-dimensional shallow water equations (see e.g. Reference [1]), which can be written in the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{h}(\mathbf{u})z_x \tag{1}$$

with

$$\mathbf{u} = \begin{bmatrix} z \\ h \\ hu \end{bmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} 0 \\ hu \\ hu^2 + gh^2/2 \end{bmatrix}, \quad \mathbf{h}(\mathbf{u}) = \begin{bmatrix} 0 \\ 0 \\ -gh \end{bmatrix} \tag{2}$$

Here z is the bottom topography, h the water height, u the water velocity, and g is the gravitational constant. Usually, the bottom topography is assumed to be given *a priori.*

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Contract/grant sponsor: HYKE; contract/grant number: HPRN-CT-2002-00282

Received 27 April 2004
Revised 22 July 2004
Accepted 26 July 2004

In (1), we consider $z = z(x)$ as an additional unknown and supply a trivial equation $z_t = 0$ for determining it.

Consider the Riemann initial data for the system (1), i.e.

$$\mathbf{u}(x,0) = \begin{cases} \mathbf{u}_{\mathrm{L}}, & x \leqslant 0 \\ \mathbf{u}_{\mathrm{R}}, & x > 0 \end{cases} \tag{3}$$

Our interest in the Riemann problem (1), (3) is motivated by two main factors. Firstly, understanding the structure of the solution to the Riemann problem (1), (3) is essential for constructing efficient Godunov-type methods for system (1). Secondly, the knowledge of the exact solution to (1), (3) provides valuable test cases for assessing the performance of numerical methods for system (1).

System (1) is derived by averaging the incompressible Navier–Stokes equations with free surface in a vertical direction. This averaging is done under the assumption that the vertical component of acceleration is negligible. Therefore, formally one is not allowed to take discontinuous initial data (3) for system (1). Indeed, depending on the jump in bottom topography $z_{\mathrm{R}} - z_{\mathrm{L}}$, one can get a large vertical component of acceleration. However, for the numerical solution to (1) one needs to discretize it, and therefore one is forced to consider the Riemann problem (1), (3) at each cell interface.

One of the main difficulties concerning system (1) is the fact that it cannot be written in divergence form, i.e. it is *non-conservative*. As a consequence, one cannot define a weak solution how it is done in the theory of conservation laws, see e.g. Reference [2]. Also, there is no analogue of the Lax–Wendroff theorem for system (1), i.e. a convergent numerical solution to (1) will not necessarily be the correct one. Therefore, there is no fundamental guideline how to construct numerical schemes for system (1), analogous to the conservation requirement for conservation laws. In this light one strives to ensure several desirable properties of a numerical method. These properties include the ability of the method to solve the steady-state solutions to (1) exactly, to be positively conservative with respect to the water height h, and be able to handle dry states $h = 0$. Some other properties can be found in e.g. Reference [3].

Typically, the quality of numerical schemes for the shallow water equations (1) is assessed on either steady-state solutions of (1), or on the problems (1) with constant bottom topography, see e.g. References [4–6]. The main goal of this work is to test the performance of different numerical methods for system (1) on the exact (unsteady) solutions to the Riemann problem (1), (3). In case of conservation laws, such assessment is a valuable criterion for designing efficient numerical methods, see e.g. Reference [7].

We use two methods for obtaining the exact solutions to the Riemann problem (1), (3), the so-called *inverse* solution, and an exact Riemann solver. The first method consists of prescribing the exact solution to the Riemann problem and determining the initial data, which correspond to this solution. This procedure is implemented in CONSTRUCT [8]. An advantage of this method is that we can easily obtain a wave configuration we are interested in, e.g. with coinciding waves, almost dry states, etc. Alternatively, one can use an exact solver for the Riemann problem (1), (3). In this work, we use the exact Riemann solver of Chinnayya *et al.* [9].

It appears that the solution to the Riemann problem (1), (3) is in general non-unique. It is not clear which of the non-unique solutions will be picked out by the numerical solution and if the numerical solution will converge towards this exact solution as the mesh is refined. In Section 4 we show that for the same Riemann initial data (3), certain methods can pick out *different* exact solutions. We also provide some ideas how one can distinguish the 'correct physical' solution.

The paper is organized as follows. In Section 2 we discuss several properties of the system of shallow water equations (1) and of the associated Riemann solution. In Section 3 we describe the procedure of obtaining exact solutions to the Riemann problem (1), (3). Section 4 contains the results of several numerical schemes, proposed for system (1). We consider the hydrostatic reconstruction method of Audusse *et al.* [4], the relaxation method of Bouchut [3], the VFRoe method of Gallouët *et al.* [5], and the kinetic method of Perthame and Simeoni [6]. We end up with conclusions in Section 5.

2. PROPERTIES OF THE RIEMANN SOLUTION

In order to provide the characteristic analysis of system (1), we rewrite it as follows:

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x = 0 \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ gh & gh - u^2 & 2u \end{bmatrix} \tag{5}$$

The eigenvalues of $\mathbf{A}(\mathbf{u})$ are

$$\lambda_0 = 0, \quad \lambda_1 = u - \sqrt{gh}, \quad \lambda_2 = u + \sqrt{gh} \tag{6}$$

One can easily show that when $\lambda_{1,2} = \lambda_0 = 0$, i.e. $u \mp \sqrt{gh} = 0$, the corresponding eigenvectors become linearly dependent. Therefore, system (1) is hyperbolic away from the critical points in the flow with $u \mp \sqrt{gh} = 0$, where it becomes parabolic degenerate.

In the solution to the Riemann problem (1), (3), each characteristic field associated with (6) can either be a shock, a rarefaction, or a contact wave, see e.g. Reference [2]. The stationary 0-contact wave plays a special role in the solution to the Riemann problem (1), (3). Indeed, the non-conservative term $\mathbf{h}(\mathbf{u})z_x$ acts only across this wave. The Riemann invariants for the 0-wave are

$$\begin{aligned} hu &= \text{const} \\ \frac{u^2}{2} + g(h+z) &= \text{const} \end{aligned} \tag{7}$$

Note that these relations are exactly the time-independent solutions of the shallow water equations (1).

Classically, each characteristic field determines a corresponding wave in the solution to the Riemann problem, separating the constant states, see e.g. Reference [2]. However, one can point out the initial data (3) for which the solution to the Riemann problem with classical waves only does not exist. In order to get existence for such initial data, one has to use certain *composite* waves, see References [9–11] for details.

System (1) belongs to the class of *resonant* systems, introduced by Isaacson and Temple [12, 13]. The Riemann problem for such systems was studied e.g. in References [10, 11, 14]. Analogous to the analysis of References [10, 11, 14] one can show that the solution to the Riemann problem (1), (3) is not unique. In Section 4 below we provide an example of the Riemann problem with the non-unique solution.

The choice of the physically relevant solution can be motivated by the comparison with the 2D or 3D incompressible free-surface code, analogously to how it is done in Reference [14]. One can consider the 2D or 3D initial data, corresponding to the 1D data, which produce non-unique solutions to the Riemann problem, and compare the 1D non-unique results with 2D or 3D averaged solution. This work is currently in progress.

3. EXACT SOLUTION TO THE RIEMANN PROBLEM

In this paper, we use the exact solutions to the Riemann problem (1), (3) in order to provide test cases for numerical methods for the shallow water equations (1). In these test cases, one is typically interested in particular flow configurations, which may be difficult to solve numerically. The common requirements to a numerical method for (1) are its ability to solve the steady-state solutions (7) exactly, and to handle the dry states $h=0$. Since system (1) is non-strictly hyperbolic, and the solution to the Riemann problem (1), (3) can be non-unique, one wishes to assess the performance of numerical methods in these cases, too.

An easy way to obtain a Riemann solution with certain properties is to solve the so-called *inverse* Riemann problem. It consists of prescribing the solution to the Riemann problem, and finding the corresponding initial data. This procedure is implemented in a software package CONSTRUCT [8]. With its help, one can easily obtain a Riemann problem with desired properties. Currently, CONSTRUCT handles only the Riemann solutions consisting of classical waves. We have used the exact Riemann solver of Reference [9] to find the solution in presence of composite waves.

4. NUMERICAL EXAMPLES

Here we present only the results on one test problem with a non-unique solution. More results can be found in the report [15]. The initial data are

$$(z,h,u) = \begin{cases} (1.5, 1.3, -2), & x \leqslant x_0 \\ (1.1, 0.1, -2), & x > x_0 \end{cases} \tag{8}$$

with $x \in [0,1]$ and $x_0 = 0.5$. One can check that the Riemann problem (1), (8) with the gravitational constant $g=2$ has two solutions. A judgement which of these solutions is physical can be done by comparison with averaged 2D or 3D incompressible Navier–Stokes equations with free surface in spirit of Reference [14].

Below we assess the performance of several numerical methods for the shallow water equations (1) on this test case. For simplicity, we consider the first-order methods. The results

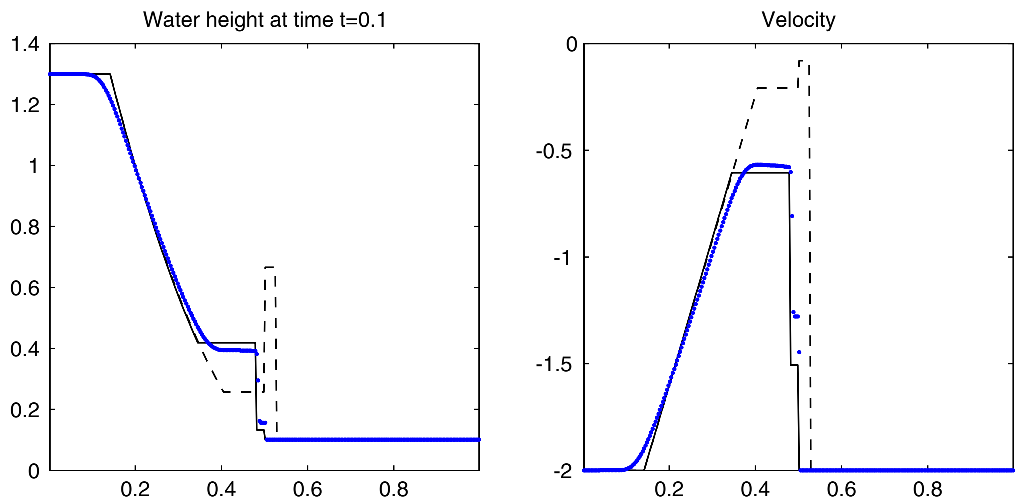

Figure 1. The results of the relaxation method [3] for the test (8).

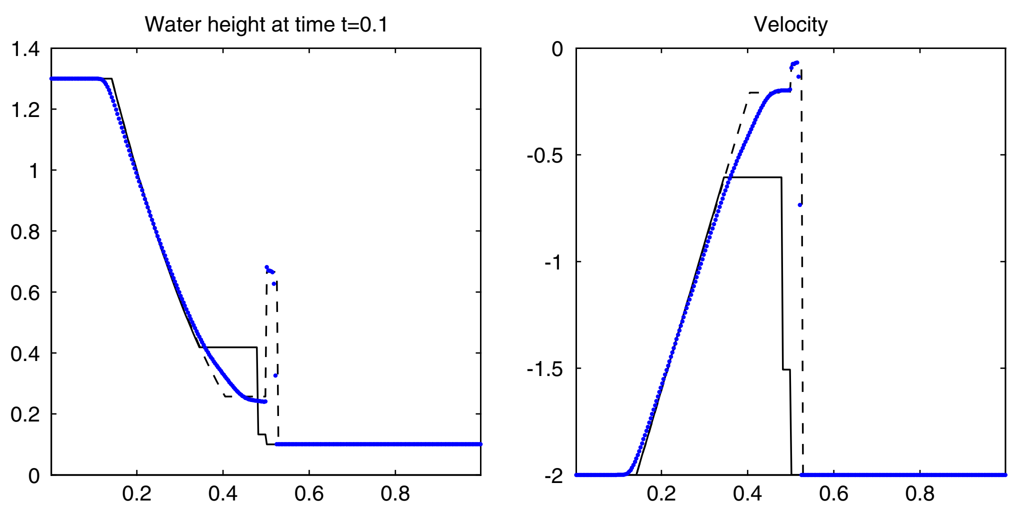

Figure 2. The results of the relaxation method with hydrostatic reconstruction [5] for the test (8).

were obtained on 300 mesh points with a CFL number of 0.9. In Figures 1 and 2, the numerical results are marked with dots, and the two non-unique exact solutions with solid and dashed lines.

Hydrostatic reconstruction method. In the hydrostatic reconstruction method [4], the first step is to compute the numerical flux of some scheme for the homogeneous shallow water equations (1), i.e. without the non-conservative term $\mathbf{h}(\mathbf{u})z_x$. To this end, we use the VFRoe [5], relaxation [3], and kinetic [6] numerical fluxes. Then, the numerical flux is modified in a way that the rest steady states of (1), i.e. relations (7) with $u=0$, are preserved on discrete level.

Relaxation solver. In the framework of the relaxation method [3], the system of shallow water equations is replaced by a modification of Suliciu's relaxation system (see e.g. Reference [16]).

The remarkable difference between the relaxation method and relaxation method with hydrostatic reconstruction [4] is that they pick out *different* exact solutions for the non-unique test (8), cf. Figures 1 and 2. Also, observe a gap between the chosen exact solution and the numerical one, which does not disappear as the mesh is refined. This means that the numerical solution *does not* converge to the exact one. The reason for such behaviour is the lack of the analogue of the Lax–Wendroff theorem for the non-conservative system (1).

VFRoe method. The VFRoe method [5] uses an approximate solution to the Riemann problem (1), (3). One can CONSTRUCT [8] a test case such that the VFRoe method produces negative water heights, see Reference [15] for an example. Sometimes one can cure this problem by diminishing the CFL number. Generally, the results of VFRoe are more oscillatory as that of the relaxation method. As for the relaxation method, the numerical solution of VFRoe with and without the hydrostatic reconstruction [4] picks out different exact solutions to the Riemann problem (1), (8).

Kinetic solver. In the kinetic approach [6] one solves a kinetic equation for the particle density, constructed in such a way that the moments of this equation are exactly the shallow water equations (1). The numerical results are similar to those of the relaxation solver [3]. Again, we use the kinetic method and the kinetic method with hydrostatic reconstruction [4] picks out different numerical solutions for the non-unique test (8).

5. CONCLUSIONS

Exact solutions to the Riemann problem for the shallow water equations provide valuable test cases for numerical methods. It appears that the solution to the Riemann problem is in general non-unique and it is not clear which exact solution will be picked out by a numerical one. We show that several numerical methods for shallow water equations can apparently pick out different exact solutions for the same initial data. Moreover, grid convergence studies show that the numerical solutions do not necessarily converge towards an exact solution.

ACKNOWLEDGEMENT

The author is grateful to F. Bouchut for providing the source codes of References [3, 5, 6] and for several helpful discussions.

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